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FEATURES OF THE PHASE-FREQUENCY CHARACTERISTICS OF THE SIMPLIFIED CIRCUIT MODEL OF AN INDUCTION COIL WITH A FERROMAGNETIC CORE

Abstract. *The paper is devoted to the study of phase-frequency characteristics of an inductance coil with a ferromagnetic core. Such a coil, unlike a coil without a core, has significantly nonlinear properties associated with active energy losses due to heating of the core by eddy currents, hysteresis losses and reactive energy leaks to the magnetic leakage flux. The nonlinearity characters are described by many parameters, which leads to difficulties in finding exact solutions to the obtained differential equations. In some cases, when a sinusoidal voltage is applied to the coil, in approximate solutions the current passing through the coil is replaced by its effective value, for which a simplified linear equivalent circuit is constructed. We study one of such widespread simplified equivalent circuits. The parameters of the special points of the phase-frequency characteristic are found.*

Keywords: *coil with ferromagnetic core, phase-frequency characteristic, simplified modelling circuit, frequency-domain simulation.*

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ОСОБЛИВОСТІ ФАЗОЧАСТОТНИХ ХАРАКТЕРИСТИК СПРОЩЕНОЇ СХЕМИ ЗАМІЩЕННЯ ІНДУКЦІЙНОЇ КОТУШКИ З ФЕРОМАГНІТНИМ СЕРДЕЧНИКОМ

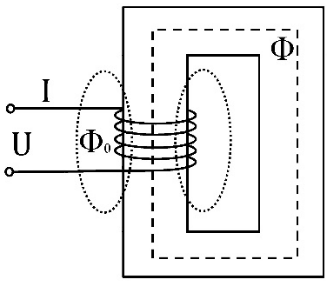
Анотація*. Робота присвячена дослідженню фазово-частотних характеристик котушки індуктивності з феромагнітним осердям. Така котушка, на відміну від котушки без сердечника, має суттєво нелінійні властивості, пов'язані з втратами активної енергії на нагрівання сердечника вихровими струмами, втратами на гістерезис та витоками реактивної енергії на магнітний потік розсіювання. Характери нелінійності описуються багатьма параметрами, що призводить до складнощів у знаходженні точних розв'язків отриманих диференціальних рівнянь. У деяких випадках при дії на котушку синусоїдальною напругою в наближених рішеннях струм, що проходить через котушку, замінюють його діючим значенням, для якого будується спрощена лінійна схема заміщення. Ми досліджуємо одну з таких поширених схем заміщення. Знайдено параметри особливих точок фазово-частотної характеристики цієї схеми заміщення.*

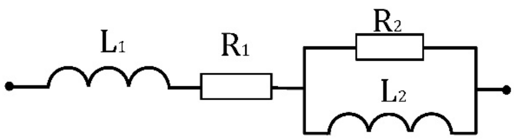
Ключові слова: *котушки індуктивності з феромагнітним осердям, фазочастотна характеристика, спрощена схема заміщення.*

Introduction. Statement of the problem. A coil placed on a ferromagnetic core is the most common element of alternating current electromagnetic devices. Its properties, to one degree or another, are characteristic of all electromagnetic devices, which consist of coils (windings) and a magnetic wire. Such devices include electromagnetic relays, transformers, magnetic amplifiers, electric machines, and others.

If a coil made of copper wire is placed in the air or on a frame made of non-ferromagnetic material, then it will be an element in an alternating current circuit with parameters R (the coil heats up when current flows through it) and L (the coil creates a magnetic flux Φ in the air. The equivalent circuit of such a coil is a series connection of ideal resistance and inductance.

If the same coil is placed on a closed ferromagnetic core (Fig. 1), its properties change significantly. As before when leaking in the alternating current coil, it creates a small magnetic flux $\mathbf{\Phi_0}$, the magnetic lines of which are closed mainly in the air. This flow is called a dissipation flow, because it does not participate in the electromagnetic transfer of energy to other coils (windings) located on the same magnetic line. At the same time, the coil creates a large magnetic flux $\mathbf{\Phi}$ in the magnetic circuit. Due to the fact that the relative magnetic permeability of ferromagnetic materials is large, the magnetic flux in the ferromagnetic conductor $\mathbf{\Phi}$ will be much greater than the magnetic leakage flux $\mathbf{\Phi_0}$.





Source: compiled by the author based on [1–5].

Fig. 1. **Inductance coil** *Fig. 2.* **Equivalent circuit**

If the active resistance of the wires of the winding $\mathbf{R_1}$ and the inductive dissipation resistance of the coil $\mathbf{L_1}$ are taken outside, then the so-called idealized coil will remain on the magnetic circuit, the properties of which depend on the properties of the magnetic circuit and its mode of operation. The conductors of such an idealized coil have no ohmic resistance and do not create a leakage flux. However, such a coil creates an alternating magnetic flux $\mathbf{\Phi}$ in the magnetic circuit and, accordingly, has an inductance. At the same time, the magnetic flux $\mathbf{\Phi}$ creates eddy currents in the magnetic circuit and periodically remagnetizes the core.

Therefore, the substitute circuit of the idealized coil should also have an active resistance $\mathbf{R_2}$, the losses in which should be equal to the losses in the magnetic flux. These losses are determined by the eddy currents in the steel, which determine the nonlinear current-voltage characteristic of $\mathbf{R_2}$.

In addition to eddy current losses, steel cores also have steel remagnetization losses – hysteresis losses (if the hysteresis loop is wide enough), which are also non-linear.

The main magnetic flux is not proportional to the current, as it is connected to it by a nonlinear magnetization curve – this is the second nonlinear reactive element $\mathbf{L_2}$.

Due to the presence of nonlinear elements, when connecting a sinusoidal voltage to coil Fig. 1, the passing current will be non-sinusoidal. However, when creating devices with ferromagnetic cores, they try not to use deep saturation modes, because in this case, leakage currents increase and higher harmonics appear, which reduce the operational properties of the device. This leads to the fact that the shape of real current curves differs little from the sinusoidal one. Therefore, when replacing the real current with a sinusoidal one with the same effective value, it is possible to use the methods of calculating linear circuits for a simplified equivalent circuit. Currently, the parallel equivalent circuit shown in Fig. 2 is widely used [1–5].

$\mathbf{R_1}$ is the active resistance of the coil, taking into account the heat losses in the copper wires. $\mathbf{L_1}$ is the inductive resistance caused by the magnetic leakage flux. $\mathbf{R_2}$ is the active resistance of the heat losses in the core. $\mathbf{L_2}$ is the inductance providing the main magnetization flux of the core.

The object of our study will be the equivalent circuit shown in Fig. 2. We will study its amplitude-frequency and phase-frequency characteristics. These functions depend on four parameters – **R₁**, **L₁**, **R₂**, and **L2**. We managed to reduce these functions to only two parameters using a suitable change of variables, which allowed us to analyse them depending on the input parameters of the equivalent circuit. At the same time, the amplitude-frequency characteristic turned out to be of little interest – it is a monotonically increasing function of frequency. On the contrary, the phase-frequency characteristic has two special extreme points for some values of the initial parameters. The exact frequency values that correspond to these special values of the phase shift were found.

Amplitude-frequency characteristic. The frequency behaviour of a linear circuit is completely determined by its complex impedance. For the circuit shown in Figure 2, the complex impedance is calculated as follows [6–9]:

$$Z = R_1 + j\omega L_1 + \frac{j\omega L_2 R_2}{R_2 + j\omega L_2}. \tag{1}$$

Its real and imaginary parts are always positive:

$$Re(Z) = R_1 + \frac{\omega^2 L_2^2 R_2}{R_2^2 + \omega^2 L_2^2} > 0\,;\ Im(Z) = \omega\left(L_1 + \frac{L_2 R_2}{R_2^2 + \omega^2 L_2^2}\right) > 0\,. \tag{2}$$

The complex impedance $Z(\omega, R_1, R_2, L_1, L_2)$ depends on four parameters of the original circuit. We transform the expression for the impedance to reduce it to two unknowns, which will allow us to conduct a detailed analysis of it.

Let us denote by

$$k = \frac{L_1 + L_2}{L_1} = 1 + \frac{L_2}{L_1} \square\ 1;\ m = \frac{R_1}{R_1 + R_2} < 1\,;\ t = \omega\frac{L_2}{R_2}. \tag{3}$$

Then

$$Z(t) = \left(R_1 + R_2\right)\left(\frac{m + t^2}{1 + t^2} + jt\cdot\frac{1 - m}{k - 1}\cdot\frac{k + t^2}{1 + t^2}\right). \tag{4}$$

The constant factor $(R_1 + R_2)$ does not depend on the frequency, so further we will analyse only the behaviour of the function Z from the reduced frequency *t*, depending on the parameters *k* and *m*, without taking this constant factor into account.

The impedance module determines the amplitude-frequency characteristic of the circuit. To study the impedance modulus, we introduce the following variables

$$a = \frac{R_2 L_1}{R_1 L_2} = \frac{1 - m}{m(k - 1)}\,;\ b = 1 + a + \frac{R_2}{R_1} = 1 + \frac{k(1 - m)}{m(k - 1)}. \tag{5}$$

Then the impedance can be written as

$$Z = R_1\frac{(1 - at^2) + jbt}{1 + jt}. \tag{6}$$

Then

$$|Z|^2 = R_1^2\left(a^2 t^2 + (b^2 - a^2 - 2a) + \frac{a^2 + 2a - b^2 + 1}{t^2 + 1}\right),\ |Z(0)| = R_1. \tag{7}$$

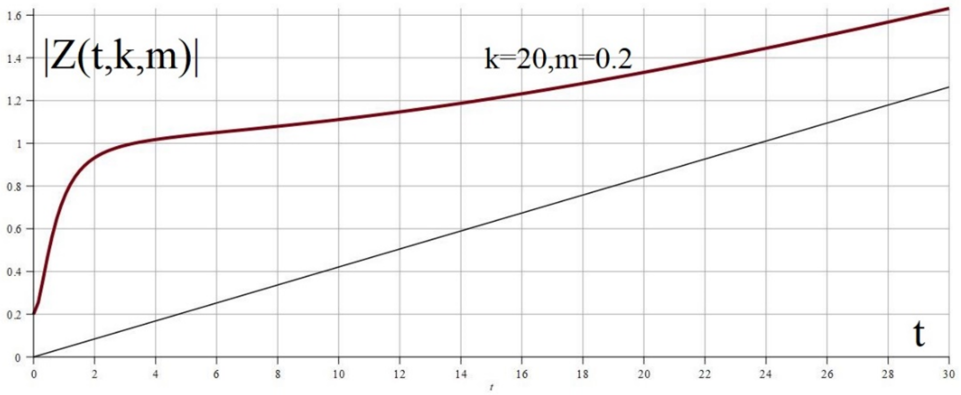
For $t \to \infty$, the impedance modulus asymptotically approaches a straight line.

$$\left|Z\right| \approx R_1 a t = \frac{R_2 L_1}{L_2} t = \frac{1-m}{k-1} t = L_1 \omega . \tag{8}$$

The physical meaning of this result is that at high frequencies the main resistance to the passing current is provided by the series-connected loss inductance due to the magnetic leakage flux.

To find the extreme points of the impedance module, we equate its derivative to zero. We get the following equation: $a^2 t^4 + 2a^2 t^2 + (b^2 - 2a - 1) = 0$. Since $a+1<b$ this equation has no roots. This means that the impedance modulus has no extreme points and increases everywhere.

A typical graph of the impedance module (amplitude-frequency characteristic of the circuit) looks like the one shown in Figure 3.



Source: author's development using a computer program.

Fig. 3. **Amplitude-frequency characteristic of the circuit**

Phase-frequency characteristic. To analyse the frequency dependence of the phase shift between the voltage and the current passing through the circuit, we will consider the tangent of the argument of the complex impedance of the circuit:

$$\tan\phi = \frac{Im\left(Z(t)\right)}{Re\left(Z(t)\right)} = \frac{1-m}{k-1} \cdot \frac{t^3 + kt}{t^2 + m} = \frac{R_2}{R_1 + R_2} \cdot \frac{L_1}{L_2}\left(t + \frac{(k-m)t}{t^2+m}\right);\ \tan\phi(0) = 0 . \tag{9}$$

For $t \to \infty$, the $\tan\phi$ asymptotically approaches a straight line with equation

$$\tan\phi(t) \approx \frac{R_2}{R_1+R_2} \cdot \frac{L_1}{L_2} \cdot t = \frac{L_1}{R_1+R_2}\omega . \tag{10}$$

The fact is that at high frequencies the phase shift is almost independent of the main inductance $\mathbf{L_2}$ – the main magnetic flux passing through the coil core.

The extreme points of $\varphi(t)$ coincide with the extreme points of $\tan(\varphi(t))$. To find them, we differentiate the function $\tan(\varphi(t))$ and find its roots.

$$\frac{d}{dt}\tan(\phi(t)) = \frac{R_2}{R_1+R_2} \cdot \frac{L_1}{L_2}\left(1 + (k-m)\frac{1-t^2}{(1+t^2)^2}\right); \tag{11}$$

$$\frac{d}{dt}\tan(\phi(t)) = 0 \Rightarrow (t^2+m)^2 = (k-m)(t^2-m) \cdot \tag{12}$$

The resulting equation has the following roots:

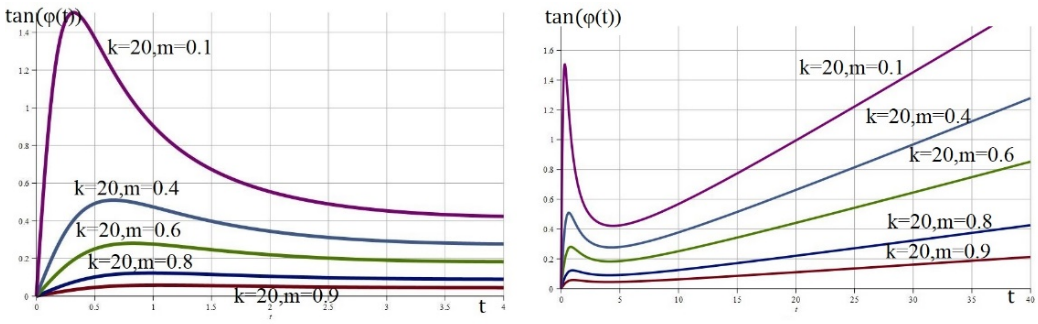
$$t^2 = \frac{1}{2}\left(k - 3m \pm \sqrt{(k-m)(k-9m)}\right) \cdot \tag{13}$$

From these formulas, we obtain the conditions for the existence of extreme points of the phase-frequency characteristic. For $k > 9m$ we have two extreme points. For $k < 9m$ there are no extreme points, the phase-frequency characteristic increases monotonically with increasing frequency. Note that $k > 1$, $m < 1$ and $(k-3m)^2 > (k-m)(k-9m)$.

In the initial parameters, the condition for the existence of two extreme points of the phase-frequency characteristic is written as follows:

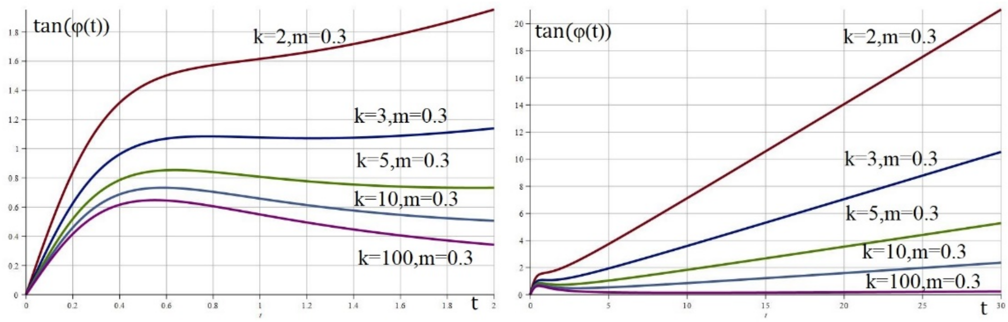
$$\frac{L_2}{L_1} > \frac{9R_1}{R_1 + R_2} - 1. \tag{14}$$

Typical graphs of the phase-frequency characteristic for some values of the parameters ***k*** and ***m*** are shown in Figures 4, 5. We prefer to use the graph of the phase shift tangent instead of the graph of the phase shift itself, since it is more informative and the asymptotes and tangent lines are clearly visible on it. Figure 4 shows how $\tan(\phi(t))$ changes with a change in ***m*** at a constant ***k***, and Figure 5 shows how $\tan(\phi(t))$ behaves at a constant ***m*** and a change in ***k***.



Source: author's development using a computer program.

Fig. 4. **Phase-frequency characteristic of the circuit at a constant *k***



Source: author's development using a computer program.

Fig. 5. **Phase-frequency characteristic of the circuit at a constant *m***

Let us now return to our main parameter ω. The extreme phase frequencies ω_1, ω_2 will be calculated using the formulas:

$$\omega^2 = \frac{1}{2} \cdot \frac{R_2^2}{L_2^2}\left(k - 3m \pm \sqrt{(k-m)(k-9m)}\right). \tag{15}$$

It seems quite natural that almost every passive element used in electrical engineering and radio engineering behaves differently at high and low frequencies. However, the definition of high and low frequency boundaries for a given element is often introduced in a purely subjective manner. For the coil under consideration, the concept of high and low frequencies can be naturally formalized. On the intervals $\omega < \omega_1$ and $\omega > \omega_2$, the behaviour of the phase and frequency has a significantly different character. While within these intervals, the behaviour of the phase and frequency is more or less uniform. The condition of the presence of two extremes for real coils is usually fulfilled – this is a reflection of the fact that the energy of the magnetic leakage flux is significantly less than the energy of the main magnetic flux inside the core.

The interval $(\omega_1;\omega_2)$ is a transition interval from low to high frequencies. Let us calculate the length and middle of this transition interval:

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R_2}{L_2}\sqrt{k-3m-2\sqrt{km}}\ ;\ \frac{\omega_2+\omega_1}{2} = \frac{1}{2}\cdot\frac{R_2}{L_2}\cdot\sqrt{k-3m+2\sqrt{km}}\ . \qquad (16)$$

(16)

Assuming that $\frac{k}{m} \gg 1$ (which is also quite common), one can write down convenient approximate values for the frequencies found:

$$t_1^2 = \frac{(k-3m)^2-(k-m)(k-9m)}{2\left(k-3m+\sqrt{(k-m)(k-9m)}\right)} = \frac{2m}{1-\frac{3m}{k}+\sqrt{\left(1-\frac{m}{k}\right)\left(1-\frac{9m}{k}\right)}} \approx m\,; \qquad (17)$$

$$t_2^2 = \frac{1}{2}k\left(1-\frac{3m}{k}+\sqrt{(1-\frac{m}{k})(1-\frac{9m}{k})}\right) \approx k\,. \qquad (18)$$

Transition interval $t \in (\sqrt{m};\sqrt{k})$. For frequency ω, we obtain an approximate interval

$$(\omega_1;\omega_2) \text{ where } \omega_1 \approx \frac{R_2}{L_2}\sqrt{\frac{R_2}{R_1+R_2}} \quad \omega_2 \approx \frac{R_2}{L_2}\sqrt{1+\frac{L_2}{L_1}}\,. \qquad (19)$$

At low frequencies, $\tan(\phi(t))$ can be approximated by a tangent line at point 0 to the graph of the function $\tan(\phi(t))$:

$$\tan(\phi(t)) \approx \tan(\gamma)\cdot t \text{ where } \tan(\gamma) = \frac{d}{dt}\tan(\phi(t))(t=0) = \frac{R_2}{R_1}\cdot\frac{L_1+L_2}{L_2}\,. \qquad (20)$$

At high frequencies, $\tan(\phi(t))$ is also approximated by a line

$$\tan(\phi(t)) \approx \tan(\delta)\cdot t \text{ where } \tan(\delta) = \frac{d}{dt}\tan(\phi(t))(t=\infty) = \frac{R_2}{R_1+R_2}\cdot\frac{L_1}{L_2}\,. \qquad (21)$$

Conclusion. As a result of the study, hitherto unknown the exact formulas for special frequencies of the phase-frequency characteristic of an elementary simplified equivalent circuit of an inductance coil with a ferromagnetic core are found (formula 15). The applicability intervals of low frequency and high frequency approximations of the parameters of such a coil are found (formulas 16, 19). The behaviour of the phase shift in the transition interval is described (Fig. 4, 5). Typical graphs of the amplitude-frequency and phase-frequency characteristics of such an equivalent circuit are given (Fig. 3, 4).

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